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## Graviton in a Curved Space-Time Background and Gauge Symmetry

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Pauli-Fierz approach to description of a massless spin-2 particle is investigated in the framework of 30-component first order relativistic wave equation theory on a curved space-time background. It is shown that additional gauge symmetry of massless equations established by Pauli-Fierz can be extended only to curved space-time regions where Ricci tensor vanishes. In all such space-time models the generally covariant S=2 massless wave equation exhibits gauge symmetry property, otherwise it is not so.

## 1 Introduction

The theory of the massive spin-2 field has received much attention over the years since the initial construction of Lagrangian formulation by Fierz and Pauli [1-2]. The original Fierz-Pauli theory for spin was second order in derivatives  $\partial_\alpha$  (and involved scalar and tensor auxiliary fields). It is highly satisfactory as long as we restrict ourselves to a free particle case. However this approach turned out not to be so good at considering spin-2 theory in presence of an external electromagnetic field. Federbush [3] showed that to avoid a loss of constraints problem the minimal coupling had to be supplemented by a direct non-minimal to the electromagnetic field strength. There followed a number of works on modification or generalizations of the Fierz-Pauli theory (Rivers [4], Nath [5], Bhargava and Watanabe [6], Tait [7], Reilly [8]). At the same time interest in general high-spin fields was generated by the discovery of the now well-known inconsistency problems of Johnson and Sudarshan [9] and Velo and Zwanzinger [10]. In the course of investigating their acausality problems for other than 3/2, Velo-Zwanzinger rediscovered the spin-2 loss of constraints problem, but were not at first aware of the non-minimal term solution of it. Velo [11] later made a thorough analysis of the external field problem for the 'correct' non-minimally coupled spin-2 theory, showing that it is acausal too.

All the work mentioned above dealt with a second-order formalism for the spin-2 theory. Much of the confusion which arose over this theory could be traced to the so-called "derivative ordering ambiguity (Naglal [12]). This problem can be avoided by working from the start with

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a first-order formalism (for example see Gel'fand et al [13]) and for which the minimal coupling procedure is unambiguous..

The work by Fedorov [14] was likely to be the first one where consistent investigation of the spin-2 theory in the framework of first-order theory was carried out in detail. The 30-component wave equation [14] referred to the so-called canonical basis, transition from which to the more familiar tensor formulation is possible but laborious task and it was not done in [14]. Subsequently the same 30-component theory was rediscovered and fundamentally elaborated in tensor-based approach by a number of authors (Regge [15], Schwinger [16], Chang [17], Hagen [18], Mathews et al [19], Cox [20]). Also a matrix formalism for the spin-2 theory was developed (Fedorov, Bogush, Krylov, Kisel [21-25]).

Concurrently else one theory for spin-2 particle was advanced that requires 50 field components (Adler [26], Deser et al [27], Fedorov and Krylov [28, 23], Cox [20.]). It appears to be more complicated, however some evident correlation between the corresponding massless theory and the non-linear gravitational equation is revealed (Fedorov [28]).

Possible connections between two variants of spin-2 theories have been investigated. Seemingly, the most clarity was achieved by Bogush and Kisel [25], who showed that 50-component equation in presence of an external electromagnetic field can be reduced to 30-component equation with additional interaction that must be interpreted as anomalous magnetic momentum term.

In the present work the 30-component first order theory is investigated in the case of vanishing mass of the particle and external curved space-time background.

## 2 Particle in the flat space-time

A system of first order wave equations describing a massless spin-2 particle in a flat space-time has the form

$$\partial^a \Phi_a = 0 , \quad (1)$$

$$\frac{1}{2} \partial_a \Phi - \frac{1}{3} \partial^b \Phi_{ab} = \Phi_a , \quad (2)$$

$$\frac{1}{2} (\partial^k \Phi_{kab} + \partial^k \Phi_{kba} - \frac{1}{2} g_{ab} \partial^k \Phi_{kn}{}^n) + \partial_a \Phi_b + \partial_b \Phi_a - \frac{1}{2} g_{ab} \partial^k \Phi_k = 0 , \quad (3)$$

$$\partial_a \Phi_{bc} - \partial_b \Phi_{ac} + \frac{1}{3} (g_{bc} \partial^k \Phi_{ak} - g_{ac} \partial^k \Phi_{bk}) = \Phi_{abc} . \quad (4)$$

A 30-component wave function consists of a scalar  $\Phi$ , vector  $\Phi_a$ , symmetric 2-rank tensor  $\Phi_{ab}$ , and 3-rank tensor  $\Phi_{abc}$  antisymmetric in two first indices. From (4) it follows four conditions that are satisfied by the 3-index field:

$$\Phi_{abc} + \Phi_{bca} + \Phi_{cab} = 0 \text{ or } \epsilon^{kabc} \Phi_{abc} = 0 . \quad (5)$$

Simplifying Eq. (4) in indices  $b$  and  $c$ , one produces

$$\partial_a \Phi_b{}^b = \Phi_{ac}{}^c . \quad (6)$$

Thus, a total number of independent components entering the theory equals 31 (instead of 30 in massive case):

$$\begin{aligned}\Phi(x) &\implies 1, & \Phi_a &\implies 4, & \Phi_{ab} &\implies 10, \\ \Phi_{abc} &\implies 6 \times 4 - 4 - 4 = 16.\end{aligned}$$

After excluding fields  $\Phi_a$  and  $\Phi_{kab}$  from (1-4) one gets to a pair of second order equations on fields  $\Phi(x)$  and  $\Phi_{ab}(x)$ <sup>1</sup>:

$$\frac{1}{2}\nabla^2\Phi - \frac{1}{3}\partial^k\partial^l\Phi_{kl} = 0, \quad (7)$$

$$(\partial_a\partial_b - \frac{1}{4}g_{ab}\nabla^2)\Phi - \frac{1}{4}g_{ab}\nabla^2\Phi_c^c + \nabla^2\Phi_{ab} - \partial_a\partial^l\Phi_{bl} - \partial_b\partial^l\Phi_{al} + \frac{1}{2}g_{ab}\partial^k\partial^l\Phi_{kl} = 0. \quad (8)$$

Allowing for (7), Eq. (8) can be rewritten as

$$(\partial_a\partial_b + \frac{1}{2}g_{ab}\nabla^2)\Phi - \frac{1}{4}g_{ab}\nabla^2\Phi_c^c + \nabla^2\Phi_{ab} - \partial_a\partial^l\Phi_{bl} - \partial_b\partial^l\Phi_{al} = 0. \quad (9)$$

The fact of prime significance in the theory under consideration is that these equations permit specific gauge principle<sup>2</sup>. That means the following: the above second order system (9) is satisfied by a a substitution (class of trivial or gradient-like solution)

$$\Phi^{(0)} = \partial^l\Lambda_l, \Phi_{ab}^{(0)} = \partial_a\Lambda_b + \partial_b\Lambda_a - \frac{1}{2}g_{ab}\partial^l\Lambda_l, \quad (10)$$

at any 4-vector function  $\Lambda_a(x)$ . Indeed,

$$-\frac{1}{3}\partial^a\partial^b\Phi_{ab}^{(0)} = -\frac{1}{2}\nabla^2\partial^l\Lambda_l = -\frac{1}{2}\nabla^2\Phi^{(0)}, \quad (11)$$

and therefore the set (10) turns Eq. (7) into identity. Further, taking into account

$$\begin{aligned}\frac{1}{2}(\partial^k\Phi_{kab} + \partial^k\Phi_{kba} - \frac{1}{2}g_{ab}\partial^k\Phi_{kn}^n) &= +\frac{1}{3}\partial^l\partial_l(\partial_b\Lambda_a + \partial_a\Lambda_b) - \frac{2}{3}\partial_a\partial_b\partial^l\Lambda_l, \\ \partial_a\Phi_b + \partial_b\Phi_a - \frac{1}{2}g_{ab}\partial^k\Phi_k &= -\frac{1}{3}\partial^l\partial_l(\partial_b\Lambda_a + \partial_a\Lambda_b) + \frac{2}{3}\partial_a\partial_b\partial^l\Lambda_l,\end{aligned}$$

one can verify that the set (10) satisfies Eq. (8) as well.

So, a massless spin-2 field in Minkowski space-time can be described by the first order system, or by the second order system (Pauli-Fierz [1-2]). At this their solutions are not determined uniquely; in general, to any chosen one we may add an arbitrary  $\Lambda_a$ -dependent term.

### 3 Particle in curved space-time

With the use of principle of minimal coupling to a curved space-time background (external gravitational field), expected generally covariant equations for a spin-2 particle are to be taken

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<sup>1</sup> The notation  $\nabla^2 = \partial^a\partial_a$  is used.

<sup>2</sup>The fact was firstly established by Pauli and Fierz [1-2].

in the form

$$\nabla^\alpha \Phi_\alpha = 0, \quad (12)$$

$$\frac{1}{2} \nabla_\alpha \Phi - \frac{1}{3} \nabla^\beta \Phi_{\alpha\beta} = \Phi_\alpha, \quad (13)$$

$$\frac{1}{2} \left( \nabla^\rho \Phi_{\rho\alpha\beta} + \nabla^\rho \Phi_{\rho\beta\alpha} - \frac{1}{2} g_{\alpha\beta}(x) \nabla^\rho \Phi_{\rho\sigma}{}^\sigma \right) + \left( \nabla_\alpha \Phi_\beta + \nabla_\beta \Phi_\alpha - \frac{1}{2} g_{\alpha\beta}(x) \nabla^\rho \Phi_\rho \right) = 0, \quad (14)$$

$$\nabla_\alpha \Phi_{\beta\sigma} - \nabla_\beta \Phi_{\alpha\sigma} + \frac{1}{3} (g_{\beta\sigma}(x) \nabla^\rho \Phi_{\alpha\rho} - g_{\alpha\sigma}(x) \nabla^\rho \Phi_{\beta\rho}) = \Phi_{\alpha\beta\sigma}. \quad (15)$$

Here  $\nabla_\alpha$  designates a generally covariant derivative. As in the flat space-time, the system exhibits the property

$$\nabla_\alpha \Phi_\beta{}^\beta = \Phi_{\alpha\beta}{}^\beta. \quad (16)$$

Now we are to investigate the question of possible gauge symmetry of the system. To this end we will try to satisfy these equations by a substitution

$$\begin{aligned} \Phi^{(0)} &= \nabla^\beta \Lambda_\beta, \\ \Phi_{\alpha\beta}^{(0)} &= \nabla_\alpha \Lambda_\beta + \nabla_\beta \Lambda_\alpha - \frac{1}{2} g_{\alpha\beta}(x) \nabla^\sigma \Lambda_\sigma, \end{aligned} \quad (17)$$

where  $\Lambda(x)$  is an arbitrary 4-vector function. With the use of Eq. (13), a vector field corresponding to the set (17) takes the form

$$\Phi_\alpha^{(0)} = \frac{2}{3} \nabla_\alpha \nabla^\beta \Lambda_\beta - \frac{1}{3} \nabla^\beta \nabla_\alpha \Lambda_\beta - \frac{1}{3} (\nabla^\beta \nabla_\beta) \Lambda_\alpha. \quad (18)$$

After substitution it into Eq. (12) one produces

$$0 = \frac{2}{3} (\nabla^\alpha \nabla_\alpha) \nabla^\beta \Lambda_\beta - \frac{1}{3} \nabla^\alpha \nabla^\beta \nabla_\alpha \Lambda_\beta - \frac{1}{3} \nabla^\beta (\nabla^\alpha \nabla_\alpha) \Lambda_\beta. \quad (19)$$

Employing conventionally the Riemann and Ricci tensors

$$(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) \Lambda_\rho = R_{\beta\alpha\rho\sigma} \Lambda^\sigma, \quad R_{\beta\alpha\dots\sigma}{}^\beta = R_{\alpha\sigma},$$

the second term in (19) can be rewritten as

$$-\frac{1}{3} \nabla^\alpha \nabla_\beta \nabla_\alpha \Lambda^\beta = -\frac{1}{3} \nabla^\alpha (\nabla_\alpha \nabla_\beta \Lambda^\beta + R_{\alpha\beta} \Lambda^\beta),$$

with the use of which Eq. (19) will take the form

$$0 = \frac{1}{3} [\nabla^\alpha \nabla_\alpha, \nabla^\beta]_- \Lambda_\beta - \frac{1}{3} \nabla^\alpha (R_{\alpha\beta} \Lambda^\beta). \quad (20)$$

The latter, with the commutator

$$[\nabla^\alpha \nabla_\alpha, \nabla^\beta]_- \Lambda_\beta = -\nabla^\alpha (R_{\alpha\sigma} \Lambda^\sigma), \quad (21)$$

will read as

$$0 = -\frac{2}{3}\nabla^\alpha(R_{\alpha\beta}\Lambda^\beta). \quad (22)$$

This equation means: if  $R_{\alpha\beta} \neq 0$ , the present spin-2 particle equations do not have any trivial  $\lambda_a$ -based solution. In other terms, a gauge principle in accordance with Einstein gravitational equations the equality  $R_{\alpha\beta} \neq 0$  speaks that at those  $x^\alpha$ -points any material fields vanish. However, in  $(R_{\alpha\beta} = 0)$  -region the wave equation under consideration includes such  $\lambda_\alpha$ -based solutions and correspondingly a gauge principle. Now, analogously, we should consider Eq. (14): what will we have had on substituting  $\Lambda_\alpha$ -set into it. We must exclude all auxiliary fields from Eq. (14):

$$\begin{aligned} & \frac{1}{2}(\nabla^\rho\Phi_{\rho\alpha\beta}^{(0)} + \nabla^\rho\Phi_{\rho\beta\alpha}^{(0)} - \frac{1}{2}g_{\alpha\beta}(x)\nabla^\rho\Phi_{\rho\sigma}^{(0)} \quad {}^\sigma) \\ & + \nabla_\alpha\Phi_\beta^{(0)} + \nabla_\beta\Phi_\alpha^{(0)} - \frac{1}{2}g_{\alpha\beta}(x)\nabla^\rho\Phi_\rho^{(0)} = 0, \end{aligned} \quad (23)$$

Let us step by step calculate all terms entering Eq. (23). For first (1) term we have that

$$\begin{aligned} (1) & \stackrel{def}{=} \frac{1}{2}\nabla^\rho\Phi_{\rho\alpha\beta}^{(0)} = \frac{1}{2}(\nabla^\rho\nabla_\rho)(\nabla_\alpha\Lambda_\beta) \\ & + \frac{1}{2}(\nabla^\rho\nabla_\rho)(\nabla_\beta\Lambda_\alpha) - \frac{1}{4}g_{\alpha\beta}(\nabla^\rho\nabla_\rho)(\nabla^\gamma\Lambda_\gamma) \\ & - \frac{1}{2}(\nabla^\rho\nabla_\rho)\nabla_\alpha\Lambda_\beta - \frac{1}{2}\nabla^\rho[\nabla_\alpha, \nabla_\rho]\Lambda_\beta \\ & - \frac{1}{2}\nabla_\alpha\nabla_\beta(\nabla^\rho\Lambda_\rho) - \frac{1}{2}[\nabla^\rho, \nabla_\alpha]\nabla_\beta\Lambda_\rho \\ & + \frac{1}{4}(\nabla_\beta\nabla_\alpha)(\nabla^\gamma\Lambda_\gamma) + \frac{1}{6}g_{\alpha\beta}(\nabla^\rho\nabla_\rho)(\nabla^\sigma\Lambda_\sigma) \\ & + \frac{1}{6}g_{\alpha\beta}\nabla^\rho[\nabla^\sigma, \nabla_\rho]\Lambda_\sigma + \frac{1}{6}g_{\alpha\beta}(\nabla^\sigma\nabla_\sigma)(\nabla^\rho\Lambda_\rho) \\ & + \frac{1}{6}g_{\alpha\beta}[\nabla^\rho, \nabla^\sigma]\nabla_\sigma\Lambda_\rho - \frac{1}{12}g_{\alpha\beta}(\nabla^\rho\nabla_\rho)(\nabla^\gamma\Lambda_\gamma) \\ & - \frac{1}{6}\nabla_\beta\nabla_\alpha(\nabla^\sigma\Lambda_\sigma) - \frac{1}{6}\nabla_\beta[\nabla^\sigma, \nabla_\alpha]\Lambda_\sigma \\ & - \frac{1}{6}(\nabla^\sigma\nabla_\sigma)\nabla_\beta\Lambda_\alpha - \frac{1}{6}[\nabla_\beta, \nabla^\sigma]\nabla_\sigma\Lambda_\alpha + \frac{1}{12}\nabla_\beta\nabla_\alpha(\nabla^\gamma\Lambda_\gamma). \end{aligned}$$

Second term in Eq. (23) can be produced on straightforward symmetry considerations from Eq. (23). Third term in Eq. (23) turns out to vanish

$$\begin{aligned} (3) & \stackrel{def}{=} -\frac{1}{4}g_{\alpha\beta}\nabla^\rho\Phi_{\rho\gamma}^{(0)} \quad {}^\gamma = -\frac{1}{2}g_{\alpha\beta}\nabla^\rho\nabla_\rho\Phi_\beta^{(0)\beta} \\ & = -\frac{1}{4}g_{\alpha\beta}\nabla^\rho\nabla_\rho(\nabla_\beta\Lambda^\beta + \nabla_\beta\Lambda^\beta - \frac{1}{2}\delta_\beta^\beta\nabla^\gamma\Lambda_\gamma) = 0. \end{aligned}$$

For fourth and fifth terms we will have

$$\begin{aligned}
(4) \stackrel{def}{=} \nabla_\alpha \Phi_\beta^{(0)} &= \frac{1}{2} \nabla_\alpha \nabla_\beta \nabla^\gamma \Lambda_\gamma - \frac{1}{3} \nabla_\alpha \nabla_\beta (\nabla^\rho \Lambda_\rho) \\
&\quad - \frac{1}{3} \nabla_\alpha [\nabla^\rho, \nabla_\beta] \Lambda_\rho - \frac{1}{3} (\nabla^\rho \nabla_\rho) \nabla_\alpha \Lambda_\beta \\
&\quad - \frac{1}{3} [\nabla_\alpha, \nabla^\rho \nabla_\rho] \Lambda_\beta + \frac{1}{6} \nabla_\alpha \nabla_\beta \nabla^\gamma \Lambda_\gamma, \\
(5) \stackrel{def}{=} \nabla_\beta \Phi_\alpha^{(0)} &= \frac{1}{2} \nabla_\beta \nabla_\alpha \nabla^\gamma \Lambda_\gamma \\
&\quad - \frac{1}{3} \nabla_\beta \nabla_\alpha (\nabla^\rho \Lambda_\rho) - \frac{1}{3} \nabla_\beta [\nabla^\rho, \nabla_\alpha] \Lambda_\rho - \frac{1}{3} (\nabla^\rho \nabla_\rho) \nabla_\beta \Lambda_\alpha \\
&\quad - \frac{1}{3} [\nabla_\beta, \nabla^\rho \nabla_\rho] \Lambda_\alpha + \frac{1}{6} \nabla_\beta \nabla_\alpha \nabla^\gamma \Lambda_\gamma;
\end{aligned}$$

and term (6) is

$$\begin{aligned}
(6) \stackrel{def}{=} & -\frac{1}{2} g_{\alpha\beta} \nabla^\rho \Phi_\rho^{(0)} - \frac{1}{2} g_{\alpha\beta} \nabla^\rho \Phi_\rho^{(0)} \\
&= -\frac{1}{4} g_{\alpha\beta} (\nabla^\rho \nabla_\rho) (\nabla^\gamma \Lambda_\gamma) + \frac{1}{6} g_{\alpha\beta} (\nabla^\rho \nabla_\rho) (\nabla^\sigma \Lambda_\sigma) \\
&\quad + \frac{1}{6} g_{\alpha\beta} \nabla^\rho [\nabla^\sigma, \nabla_\rho] \Lambda_\sigma + \frac{1}{6} g_{\alpha\beta} (\nabla^\sigma \nabla_\sigma) (\nabla^\rho \Lambda_\rho) \\
&\quad + \frac{1}{6} g_{\alpha\beta} [\nabla^\rho, \nabla^\sigma \nabla_\sigma] \Lambda_\rho - \frac{1}{12} g_{\alpha\beta} (\nabla^\rho \nabla_\rho) (\nabla^\gamma \Lambda_\gamma).
\end{aligned}$$

Summing up all six expressions and taking into account similar terms (factors at all terms without commutators turn out to be equal zero as should be expected):

$$\begin{aligned}
0 &= (\nabla^\rho \nabla_\rho) (\nabla_\alpha \Lambda_\beta) \left[ \left( \frac{1}{2} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{6} \right) - \frac{1}{3} \right] \\
&\quad + (\nabla^\rho \nabla_\rho) (\nabla_\beta \Lambda_\alpha) \left[ \left( \frac{1}{2} - \frac{1}{6} \right) + \left( \frac{1}{2} - \frac{1}{2} \right) - \frac{1}{3} \right] \\
&\quad + g_{\alpha\beta} (\nabla^\rho \nabla_\rho) (\nabla^\gamma \Lambda_\gamma) \left[ \left( -\frac{1}{4} + \frac{1}{6} + \frac{1}{6} - \frac{1}{12} \right) + \left( -\frac{1}{4} + \frac{1}{6} + \frac{1}{6} - \frac{1}{12} \right) \right] \\
&\quad + \nabla_\alpha \nabla_\beta (\nabla^\rho \Lambda_\rho) \left[ \left( -\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{12} \right) + \left( -\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{12} \right) \right. \\
&\quad \quad \left. + \left( \frac{1}{2} - \frac{1}{3} + \frac{1}{6} \right) + \left( \frac{1}{2} - \frac{1}{3} + \frac{1}{6} \right) \right] \\
&\quad + \left\{ -\frac{1}{2} \nabla^\rho [\nabla_\alpha, \nabla_\rho] \Lambda_\beta - \frac{1}{2} [\nabla^\rho, \nabla_\alpha \nabla_\beta] \Lambda_\rho \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{6} g_{\alpha\beta} \nabla^\rho [\nabla^\sigma, \nabla_\rho] - \Lambda_\sigma + \frac{1}{6} g_{\alpha\beta} [\nabla^\rho, \nabla^\sigma \nabla_\sigma] - \Lambda_\rho - \\
& \quad - \frac{1}{6} \nabla_\beta [\nabla^\sigma, \nabla_\alpha] - \Lambda_\sigma - \frac{1}{6} [\nabla_\beta, \nabla^\sigma \nabla_\sigma] - \Lambda_\alpha \Big\} \\
& + \Big\{ - \frac{1}{2} \nabla^\rho [\nabla_\beta, \nabla_\rho] - \Lambda_\alpha - \frac{1}{2} [\nabla^\rho, \nabla_\beta \nabla_\alpha] - \Lambda_\rho \\
& + \frac{1}{6} g_{\beta\alpha} \nabla^\rho [\nabla^\sigma, \nabla_\rho] - \Lambda_\sigma + \frac{1}{6} g_{\beta\alpha} [\nabla^\rho, \nabla^\sigma \nabla_\sigma] - \Lambda_\rho \\
& \quad - \frac{1}{6} \nabla_\alpha [\nabla^\sigma, \nabla_\beta] - \Lambda_\sigma - \frac{1}{6} [\nabla_\alpha, \nabla^\sigma \nabla_\sigma] - \Lambda_\beta \Big\} + \\
& + \Big\{ - \frac{1}{3} \nabla_\alpha [\nabla^\rho, \nabla_\beta] - \Lambda_\rho - \frac{1}{3} [\nabla_\alpha, \nabla^\rho \nabla_\rho] - \Lambda_\beta \Big\} \\
& + \Big\{ - \frac{1}{3} \nabla_\beta [\nabla^\rho, \nabla_\alpha] - \Lambda_\rho - \frac{1}{3} [\nabla_\beta, \nabla^\rho \nabla_\rho] - \Lambda_\alpha \Big\} \\
& + \frac{1}{6} g_{\alpha\beta} (\nabla^\rho [\nabla^\sigma, \nabla_\rho] - \Lambda_\sigma + [\nabla^\rho, \nabla^\sigma \nabla_\sigma] - \Lambda_\rho) \Big\}.
\end{aligned}$$

Calculating in series all commutators, after simple calculation we will produce

$$\begin{aligned}
0 = & g_{\alpha\beta} \nabla_\rho (R^{\rho\sigma} \Lambda_\sigma) + \Lambda^\sigma \left[ \nabla_\rho R^\rho_{\alpha\beta\sigma} + \nabla_\rho R^\rho_{\beta\alpha\sigma} \right] \\
& + (\nabla_\rho \Lambda_\sigma) \left[ R^\rho_{\alpha\beta}{}^\sigma + R^\rho_{\beta\alpha}{}^\sigma \right] \\
& - \Lambda^\rho \left[ \nabla_\alpha R_{\beta\rho} + \nabla_\beta R_{\alpha\rho} \right] - \frac{3}{2} \left[ R_\beta{}^\rho (\nabla_\alpha \Lambda_\rho) + R_\alpha{}^\rho (\nabla_\beta \Lambda_\rho) \right] \\
& + \frac{1}{2} \left[ R_\beta^\rho (\nabla_\rho \Lambda_\alpha) + R_\alpha{}^\rho (\nabla_\rho \Lambda_\beta) \right]. \tag{24}
\end{aligned}$$

It must be noticed that contrary to the expectations the equation obtained contains explicitly the curvature Riemann tensor. It enters into Eq. (24) in two combinations:

$$\Lambda^\sigma (\nabla_\rho R^\rho_{\alpha\beta\sigma} + \nabla_\rho R^\rho_{\beta\alpha\sigma}), \quad (\nabla_\rho \Lambda_\sigma) [R^\rho_{\alpha\beta}{}^\sigma + R^\rho_{\beta\alpha}{}^\sigma]. \tag{25}$$

The curvature tensor in combination (25) can be readily escaped. To this end, it suffices for the Bianchi identity

$$\nabla_\gamma R^\rho_{\alpha\beta\sigma} + \nabla_\sigma R^\rho_{\alpha\gamma\beta} + \nabla_\beta R^\rho_{\alpha\sigma\gamma} = 0, \quad \nabla_\rho R^\rho_{\alpha\beta\sigma} + \nabla_\sigma R_{\beta\alpha} - \nabla_\beta R_{\alpha\sigma} = 0.$$

Thus,

$$\nabla_\rho R^\rho_{\alpha\beta\sigma} + \nabla_\rho R^\rho_{\beta\alpha\sigma} = (\nabla_\alpha R_{\beta\sigma} + \nabla_\beta R_{\alpha\sigma}) - 2\nabla_\sigma R_{\beta\alpha}. \tag{26}$$

With Eq. (26), Eq. (24) takes the form

$$\begin{aligned}
0 = & g_{\alpha\beta} \nabla_\rho (R^{\rho\sigma} \Lambda_\sigma) - 2\Lambda^\sigma \nabla_\sigma R_{\alpha\beta} + (\nabla_\rho \Lambda_\sigma) [R^\rho_{\alpha\beta}{}^\sigma + R^\rho_{\beta\alpha}{}^\sigma] \\
& - \frac{3}{2} [R_\beta{}^\rho (\nabla_\alpha \Lambda_\rho) + R_\alpha{}^\rho (\nabla_\beta \Lambda_\rho)] + \frac{1}{2} [R_\beta^\rho (\nabla_\rho \Lambda_\alpha) + R_\alpha{}^\rho (\nabla_\rho \Lambda_\beta)]. \tag{27}
\end{aligned}$$

However, the curvature tensor still remains to enter Eq. (27). And this means that in regions involving curvature the above massless spin-2 equation does not allow any gauge principle.

Now we will show that in order to overcome such a difficulty the above starting equations should be slightly altered. To this end, let us add special term (a not minimal gravitational interaction term) into Eq. (14):

$$\begin{aligned} & \frac{1}{2} \left( \nabla^\rho \Phi_{\rho\alpha\beta} + \nabla^\rho \Phi_{\rho\beta\alpha} - \frac{1}{2} g_{\alpha\beta}(x) \nabla^\rho \Phi_{\rho\sigma}{}^\sigma \right) \\ & + \left( \nabla_\alpha \Phi_\beta + \nabla_\beta \Phi_\alpha - \frac{1}{2} g_{\alpha\beta}(x) \nabla^\rho \Phi_\rho \right) = A [R^\rho{}_{\alpha\beta}{}^\sigma + R^\rho{}_{\beta\alpha}{}^\sigma] \Phi_{\rho\sigma}. \end{aligned} \quad (28)$$

Let us show that at special parameter  $A$  the theory of massless spin-2 particle can be done satisfactory in the sense of the above gauge principle. Indeed,

$$\begin{aligned} AR^\rho{}_{\alpha\beta}{}^\sigma \Phi_{\rho\sigma}^{(0)} &= AR^\rho{}_{\alpha\beta}{}^\sigma \left( \nabla_\rho \Lambda_\sigma + \nabla_\sigma \Lambda_\rho - \frac{1}{2} g_{\rho\sigma} \nabla^\gamma \Lambda_\gamma \right) \\ &= A \left[ R^\rho{}_{\alpha\beta}{}^\sigma \nabla_\rho \Lambda_\sigma + R^\rho{}_{\beta\alpha}{}^\sigma \nabla_\rho \Lambda_\sigma + \frac{1}{2} R_{\alpha\beta} \nabla^\gamma \Lambda_\gamma \right] \end{aligned}$$

and therefore a contribution of that additional term into (27) is equal to

$$A [R^\rho{}_{\alpha\beta}{}^\sigma + R^\rho{}_{\beta\alpha}{}^\sigma] \Phi_{\rho\sigma}^{(0)} = 2A [R^\rho{}_{\alpha\beta}{}^\sigma + R^\rho{}_{\beta\alpha}{}^\sigma] (\nabla_\rho \Lambda_\sigma) + AR_{\alpha\beta} (\nabla^\gamma \Lambda_\gamma). \quad (29)$$

So, instead of Eq. (27) we have

$$\begin{aligned} & 2A (\nabla_\rho \Lambda_\sigma) [R^\rho{}_{\alpha\beta}{}^\sigma + R^\rho{}_{\beta\alpha}{}^\sigma] + AR_{\alpha\beta} (\nabla^\gamma \Lambda_\gamma) \\ &= g_{\alpha\beta} \nabla_\rho (R^{\rho\sigma} \Lambda_\sigma) - 2\Lambda^\sigma \nabla_\sigma R_{\alpha\beta} + (\nabla_\rho \Lambda_\sigma) [R^\rho{}_{\alpha\beta}{}^\sigma + R^\rho{}_{\beta\alpha}{}^\sigma] \\ & - \frac{3}{2} [R_\beta{}^\rho (\nabla_\alpha \Lambda_\rho) + R_\alpha{}^\rho (\nabla_\beta \Lambda_\rho)] + \frac{1}{2} [R_\beta{}^\rho (\nabla_\rho \Lambda_\alpha) + R_\alpha{}^\rho (\nabla_\rho \Lambda_\beta)]. \end{aligned} \quad (30)$$

Setting  $A = \frac{1}{2}$ , both terms with curvature tensor will be cancelled by each other:

$$\begin{aligned} & \frac{1}{2} R_{\alpha\beta} (\nabla^\gamma \Lambda_\gamma) = g_{\alpha\beta} \nabla_\rho (R^{\rho\sigma} \Lambda_\sigma) - 2\Lambda^\sigma \nabla_\sigma R_{\alpha\beta} \\ & - \frac{3}{2} [R_\beta{}^\rho (\nabla_\alpha \Lambda_\rho) + R_\alpha{}^\rho (\nabla_\beta \Lambda_\rho)] + \frac{1}{2} [R_\beta{}^\rho (\nabla_\rho \Lambda_\alpha) + R_\alpha{}^\rho (\nabla_\rho \Lambda_\beta)]. \end{aligned} \quad (31)$$

Finally the obtained relationship does not contain the curvature tensor and will turn into identity at  $R_{\alpha\beta}(x) = 0$  which was required. So, the required system is which one changes Eq. (14) by

$$\begin{aligned} & \frac{1}{2} \left( \nabla^\rho \Phi_{\rho\alpha\beta} + \nabla^\rho \Phi_{\rho\beta\alpha} - \frac{1}{2} g_{\alpha\beta}(x) \nabla^\rho \Phi_{\rho\sigma}{}^\sigma \right) \\ & + \left( \nabla_\alpha \Phi_\beta + \nabla_\beta \Phi_\alpha - \frac{1}{2} g_{\alpha\beta}(x) \nabla^\rho \Phi_\rho \right) = \frac{1}{2} (R^\rho{}_{\alpha\beta}{}^\sigma + R^\rho{}_{\beta\alpha}{}^\sigma) \Phi_{\rho\sigma}. \end{aligned} \quad (32)$$



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